Functions of two variables

Schwarz's Theorem :→

Statement :→ Let $f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a scalar field defined in some neighbourhood of a point $(a,b) \in \mathbb{R}^2$ such that

(i) both the first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exists finitely in some neighbourhood of the point (a,b)

(ii) $\frac{\partial^2 f}{\partial y \partial x}$ exists and is continuous in some neighbourhood of the point (a,b).

Then $\frac{\partial^2 f}{\partial x \partial y}(a,b)$ exists finitely and

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$$

Proof :→

Since the given conditions are satisfied in some neighbourhood S of the point (a,b), We can always find $h, k \in \mathbb{R}$ such that the rectangle with end points (a,b), $(a+h, b)$, $(a, b+k)$, $(a+h, b+k)$ lie entirely in the said neighbourhood S.

Let $\Delta(h,k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)$

Let us define

$$\phi(x) = f(x, b+k) - f(x, b)$$

for all $x \in [a, a+h]$ or $[a+h, b]$ depending on whether $h>0$ or $h>0$. So,

$$\Delta(h,k) = \phi(a+h) - \phi(a)$$

$\because \frac{\partial f}{\partial x}$ exists and is continuous in some neighbourhood S, ϕ' exists in S.

Applying mean value theorem,

$$\Delta(h,k) = \phi(a+h) - f(a)$$
$$= h\phi'(a+\theta_1 h), \text{ where } 0<\theta_1<1.$$
$$= h\left\{\frac{\partial f}{\partial x}(a+\theta_1 h, b+k) - \frac{\partial f}{\partial x}(a+\theta_1 h, b)\right\}$$

Applying M.V. theorem

$$= h\left\{k \frac{\partial^2 f}{\partial y \partial x}(a+\theta_1 h, b+\theta_2 k)\right\}.$$

where $0<\theta_2<1$.

Obviously the point $(a+\theta_1 h, b+\theta_2 k)$ lies inside the rectangle with (a,b) and $(a+h, b+k)$ as digonal points.

Now,

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(a+h, b) - \frac{\partial f}{\partial y}(a,b)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left\{\frac{\partial f}{\partial y}(a+h, b) - \frac{\partial f}{\partial y}(a,b)\right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{ \lim_{k\to 0} \frac{f(a+h, b+k) - f(a+h, b)}{k} - \lim_{k\to 0} \frac{f(a, b+k) - f(a,b)}{k} \right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{ \lim_{k\to 0} \left(f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b) \right) \right\}$$

$$= \lim_{h\to 0} \lim_{k\to 0} \frac{\Delta(h,k)}{h,k}$$

In order to prove the proposition, we are to show that

$$\lim_{k\to 0}\lim_{k\to 0} \frac{\partial^2 f}{\partial y \partial x}(a+\theta_1 h, b+\theta_2 k) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$$

Obviously,

$$\lim_{k\to 0} \frac{\partial^2 f}{\partial y \partial x}(a+\theta_1 h, b+\theta_2 k) \quad \text{exists.}$$

Let

$$g(h) = \lim_{k\to 0} \frac{\partial^2 f}{\partial y \partial x}(a+\theta_1 h, b+\theta_2 k)$$

So, we required to assure that

$$\lim_{h\to 0} g(h) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$$

$\because \frac{\partial^2 f}{\partial y \partial x}$ is continuous in some neighbourhood of (a,b), corresponding to $\epsilon > 0$, we can find a $\delta = \delta(\epsilon) > 0$ s.t.

$$\|(x,y)-(a,b)\| < \delta \Rightarrow \left|\frac{\partial^2 f}{\partial y \partial x}(x,y) - \frac{\partial^2 f}{\partial y \partial x}(a,b)\right| < \epsilon$$

We choose δ s.t. $(a+\theta_1 h, b+\theta_2 k) = (x,y) \in S((a,b),\delta)$.

Then keeping $k \to 0$, we find

$$\left|\lim_{k\to 0} \frac{\partial^2 f}{\partial y \partial x}(a+\theta_1 h, b+\theta_2 k) - \lim_{k \to 0} \frac{\partial^2 f}{\partial y \partial x}(a,b)\right| \leq \epsilon$$

$$\Rightarrow \left| g(h) - \frac{\partial^2 f}{\partial y \partial x}(a,b)\right| \leq \epsilon$$

$\because \epsilon$ is arbitrary, we have

$$h \to 0 \Rightarrow g(h) \to \frac{\partial^2 f}{\partial y \partial x}(a,b)$$

i,e; $\displaystyle\lim_{h\to 0} g(h) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$

This completes the proof.